Stability in Matching with Externalities: Pairs Competition and Oligopolistic Joint Ventures

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1 Introduction

matchings might be realized by forming a pair, and showed the existence of stable matching under pessimism as in Sasaki and Toda (1996). Chen (2019) considered a specific example of Cournot oligopoly game played by joint ventures, and assumed that every potential partner for a company induces a unique consistent expectation for the realized matching. With this list of expectations for each possible pair, stable matching is defined as the outcome of this game. Chen identified conditions under which positively and negatively assortative matchings are stable.

In these papers, each player has expectations on the realization of a matching when she is partnered with each of the players on the other side of the market, and stable matching is built on these expectations. In contrast, this paper mirrors the original definition of pairwise stable matching in matching problems without externalities. Our pairwise stability starts with a matching and checks whether or not there is a pair of players with a profitable deviation away from the original matching. There is, however, a subtle issue in the presence of externalities— the rest of matching matters. Thus, we need to formulate the matching induced by the deviation of a pair from the original matching. We specify this using an effectiveness function, and consider two effectiveness functions specifically. The first is that after a deviation by a pair, the dumped partners stay single and no other player changes their partners. The second is that after a deviation by a pair, the dumped partners form a pair and all other players remain with their partners. The former effectiveness function is in the literature of theory of coalition formation, and is adopted to analyze convergence of a sequence of myopic deviations in a marriage problem by Roth and Vande Vate (1990).[1] The latter effectiveness function was proposed by Knuth (1976) in the context of the marriage problem in which all players are acceptable to all other players.

To see the difference between these two effectiveness functions, consider the example of a pairs figure skating competition. Suppose that there are three male and three female skaters, one with high, medium, and low ability in each gender. Moreover, suppose that there are complementarities in partners' abilities. Then, it is natural for them to have a (positively) assortative matching, since a high ability partner is always desirable. However, this assortative matching may not be pairwise stable in the Roth-Vande Vate sense. Consider a deviation by the high ability male and the

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swapping. In Section 4, we consider an oligopolistic joint ventures problem, which is an assignment game version of the pairs competition model without endogenous efforts. We show that pairwise stable assignments can be supported only by the assortative matching, and characterize the one-side optimal stable assignment. In Section 5, we introduce personalized intrinsic utility from partners and heterogeneous match qualities of players, and show that pairwise stable assignment via swapping may not exist. Section 6 concludes.

1.1 A Brief Literature Review

There are three branches of literature that are related to the current paper. The first branch is the one of matching with externalities. Recently, a number of papers have been written in this field in addition to Sasaki and Toda (1996), Hafalir (2008), and Chen (2019). Mumcu and Saglam (2010), Fisher and Hafalir (2016), and Chade and Eeckout (2020) all dealt with one-to-one matching problems with externalities in different ways. Mumcu and Saglam (2010) introduced outside options, and Fisher and Hafalir (2016) and Chade and Eeckhout (2020) removed the impacts of pairwise deviations through externalities by imposing a behavioral assumption and by considering a continuum of atomless agents, respectively. Bando (2012, 2014), and Pycia and

(^{0}((m)) = (m)), and if w is paired under ((w) $\neq$ w) then w's partner (w) is single under ^{0} (^{0}((w)) = (w)); and (iii) for all

((m_i) 2 W), team i's members' e¤orts are aggregated by a CES function

$$Y_i = (a_m e_m + a_{(m)} e_{(m)})^{\frac{1}{}}; \qquad (1)$$

where

and player x's equilibrium e¤ort given Y_i and Y can be written as

$$e_x = Y_i \quad (1 \qquad _i)\frac{1}{Y}^{\frac{1}{1-}} a_x^{\frac{}{1-}} V^{\frac{1}{1-}}: \tag{3}$$

Raising this to the power of and then multiply it by a_x,

$$a_x e_x = Y_i \quad (1 \qquad _i)\frac{1}{Y}^{\frac{}{1-}} a_x^{\frac{}{1-}} V^{\frac{}{1-}}$$

is obtained (the power of a_x is calculated by $\frac{\;^2}{1\;} + \quad = \frac{}{1\;}$). Substituting this back to (1), we obtain

$$Y_i = Y_i \quad (1 \qquad _i)\frac{1}{Y}^{\frac{1}{1-}} \quad a_x^{\frac{}{1-}} + a_{(x)}^{\frac{}{1-}} \quad ^{\frac{1}{}} V^{\frac{1}{1-}}$$

or

$$\frac{1}{A_i(\;)} = \frac{Y \quad _i}{Y^2} V; \tag{4}$$

where $A_i(\;) = \quad a_m^{\frac{}{1-}} + a_{(m\;)}^{\frac{}{1-}} \quad ^{\frac{1-}{}}$ stands for the

stable if and only if (i) $\mu R_m \mu'$ or $\mu R_w \mu'$ for any pairwise deviations $(m, w) \in M \times W$ with $\mu \to_{(m,w)} \mu'$, and (ii) $\mu R_x \mu'$ for any single player deviation $x \in M \cup W$ with $\mu \to_x \mu'$. The following example shows that there may not be a pairwise stable matching.

Example 1. Consider a figure skating contest with $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Let $\lambda = \frac{1}{2}$, $a_{m_1} = a_{w_1} = 1$, $a_{m_2} = a_{w_2} = 0.9$, and $a_{m_3} = a_{w_3} = 0.7$. We calculate m_1's payoffs under the assortative matching and the one after he deviates with w_2:

(i) $\mu = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$:

$$U_{m_1}(\mu) = \left(1 - \frac{2^{-\frac{1}{2}}}{\frac{1}{2} + \frac{1}{1.8} + \frac{1}{1.4}}\right)\left(1 - \frac{2^{-\frac{1}{2}}}{\frac{1}{2} + \frac{1}{1.8} + \frac{1}{1.4}}\right)^{\frac{1}{2}} = 0.31209$$

(ii) $\mu' \equiv \{(m_1, w_2), (m_3, w_3)\}$:

$$U_{m_1}(\mu') = \left(1 - \frac{\frac{1}{1.9}}{\frac{1}{1.9} + \frac{1}{1.4}}\right)\left(1 - \frac{\frac{1}{1.9}}{\frac{1}{1.9} + \frac{1}{1.4}}\right)^{\frac{1}{1.9}} = 0.44720$$

Thus, agent m_1 is better off by dumping his higher ability partner for an inferior partner. A similar deviation blocks any other fully matched matching, and if agents are not fully matched in matching μ, then μ is blocked by an unmatched pair. Thus, there is no pairwise stable matching in this example.□

The problem underlying this example is that players prefer to have a smaller number of rival pairs, and the best player would rather have a weaker partner if the number of rival pairs goes down. However, since single players cannot participate in the competition, resulting in receiving the lowest payoffs, it does not make sense to expect that they will stay singles. If the single players becomes a pair, the number of rivals do not change, undermining the motivation for the best player to seek a lower ability partner. Using the second effectiveness function $\rightrightarrows_S$ allows us to define the following alternative stability concept. A matching μ is pairwise stable via swapping if and only if (i) $\mu R_m \mu'$ or $\mu R_w \mu'$ for any pairwise deviations $(m, w) \in M \times W$ with $\mu \rightrightarrows_{(m,w)} \mu'$. In the following, we will show that the assortative matching μ^* is uniquely stable in the above sense. We first prove the following lemma, which demonstrates that an assortative swapping improves higher ability players' payoffs.

Lemma 1. Let ,

Now, we borrow the model from Shubik (1984) to describe our oligopolistic market.[7] Suppose that there are n products produced by n active joint ventures together with a numeraire commodity (the 0th commodity). There is unit mass of identical consumers, each with a quadratic utility function:

$$u = \sum_{i=1}^{n} x_i - \frac{1}{2}\sum_{i=1}^{n} x_i^2 - \frac{\gamma}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} x_k x_j + x_0; \tag{5}$$

where $\gamma \in [0;1)$ is a substitution parameter between products. As γ increases, substitutability

By Lemmas 3 and 4, we know that X is strictly supermodular and strictly increasing. The following proposition shows that there are pairwise stable assignments, and characterizes the M-optimal pairwise stable matching by using the above output matrix X.[8]

Proposition 5. In an oligopolistic joint venture model, there exist pairwise stable assignments. Under the M-optimal pairwise stable assignment, the pairwise stable payoff vector for W is minimized at $s^* = (s_1^*, \ldots, s_n^*)$ where $s_j^* = \sum_{j'=j}^{n-1}(X_{j'+1j'} - X_{j'+1j'+1})$ for any $j \leq n-1$ and $s_n^* = 0$, and the stable payoff vector for M is calculated by $r_j^* = X_{jj} - s_j^*$.

Bulow and Levin (2006) derived the above simple "minimum competitive salary" formula in the context of firm-worker matching problem with output function $X_{ij} = a_i a_j$ (thus with

This payoff function is composed of two parts: $b_x(\mu(x))$ incorporates agent x's intrinsic payoff from being matched with $\mu(x)$ independent of externalities or competition outcome. As before, we assume that players decide how much effort to make after a matching μ has been determined. Since the first term enters additively, players' effort decisions depend only on the latter part of $\tilde{U}_x$. Thus, for all $\mu \in M^F$, equilibrium payoff is

$$\tilde{U}_x(\mu) = \gamma b_x(\mu(x)) + (1-\gamma) U_x(\mu);$$

where $U_x(\mu)$ is the same as in Section 3:

$$U_x(\mu) = \left[1 - \frac{(n(\mu)-1)\frac{1}{A(\mu)}}{\sum_{j=1}^{n}\frac{1}{A(\mu)}}\right]\left[1 - \frac{(n(\mu)-1)\frac{1}{A(\mu)}}{\sum_{j=1}^{n}\frac{1}{A(\mu)}}\left(\frac{a_x}{A_i(\mu)}\right)^{\frac{1}{1-\alpha}}\right] V:$$

Clearly, when $\gamma = 0$, this problem degenerates to the pairs competition problem, and to the standard one-to-one matching problem without externalities when $\gamma = 1$. In the neighborhoods of $\gamma = 1$ or $\gamma = 0$, pairwise stable matching via swapping obviously exists. But does this hold when γ is significantly far from the end points? Unfortunately, in general, pairwise stable matching via swapping may not exist as we can see from the following example.

Example 2. Consider a figure skating contest with $M = \{m_1; m_2; m_3\}$ and $W = \{w_1; w_2; w_3\}$. Let $\gamma = \frac{1}{2}$, $a_{m_1} = a_{w_1} = 0.5$, $a_{m_2} = a_{w_2} = 0.4$, and $a_{m_3} = a_{w_3} = 0.25$. Personal intrinsic payoffs from the partners are described by the following matrices:

$$\begin{pmatrix} b_{m_1}(w_1) & b_{m_1}(w_2) & b_{m_1}(w_3) \\ b_{m_2}(w_1) & b_{m_2}(w_2) & b_{m_2}(w_3) \\ b_{m_3}(w_1) & b_{m_3}(w_2) & b_{m_3}(w_3) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1.14 \end{pmatrix}$$

$$\left(b \right.$$

(ii) ${}^{2} = f(m_1; w_1); (m_2; w_3); (m_3; w_2)g$: $U_{m_1}(\ {}^{2}) = U_{w_1}(\ {}^{2}) = 0{:}384$; $U_{m_2}(\ {}^{2}) = U_{w_2}(\ {}^{2}) = 0{:}131$; $U_{m_3}(\ {}^{2}) = U_{w_3}(\ {}^{2}) = 0{:}174$.

(iii) ${}^{3} = f(m_1; w_3); (m_2; w_1); (m_3; w_2)g$: $U_{m_1}(\ {}^{3}) = 0{:}183$; $U_{m_2}(\ {}^{3}) = 0{:}332$; $U_{m_3}(\ {}^{3}) = 0{:}160$; $U_{w_1}(\ {}^{3}) = 0{:}305$; $U_{w_2}(\ {}^{3}) = 0{:}119$; $U_{w_3}(\ {}^{3}) = 0{:}257$.

m_2, and m_2 to m_1.[10] Therefore, it is not easy to assure the existence of pairwise stable matching via swapping. If, however, ability ranking agrees with intrinsic preference ranking for all agents, then we have the following result.

Proposition 6. Suppose that $b_m(w_1) \quad b_m(w_2) \quad \cdots \quad b_m(w_n)$ for all $m_i \in M$ and $b_w(m_1) \quad b_w(m_2) \quad \cdots \quad b_w(m_n)$ for all $w_i \in W$. Then, is the unique pairwise stable matching via swapping.

5.2 Match Qualities

Here, we introduce match qualities of pairs, and we introduce match qualities in the oligopolistic joint venture problem. Let a match quality matrix Q be

$$Q = \begin{pmatrix} q_{11} & \cdots & q_{1j} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ q_{i1} & \cdots & q_{ij} & \cdots & q_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nj} & \cdots & q_{nn} \end{pmatrix};$$

where $q_{ij} \in (0,1]$ which captures how good a match between m_i and w_j is. Effectively, this describes how well m_i and w_j can work together.

In this case, the structure of the problem are the same as the original oligopolistic joint ventures, so our solution concept, the pairwise stable assignment via swapping, is well defined. However, with an arbitrary match quality matrix Q, a pairwise stable assignment via swapping may not exist. This is because we can create an arbitrary matrix of the marginal cost of each joint venture, $C = (c(m_i; w_j))_{i;j=1;\ldots;n}$ by freely choosing Q. The following example demonstrates this result.

Example 3. Let $M = \{m_1; m_2; m_3\}$ and $W = \{w_1; w_2; w_3\}$ with

$$C = \begin{pmatrix} 0.1 & 0.11 & 0.3 \\ 0.3 & 0.1 & 0.3 \\ 0.3 & 0.3 & 0.3 \end{pmatrix}.$$

There are six full matchings:

(i) ${}_1(m_1) = w_1;\ {}_1(m_2) = w_2;\ {}_1(m_3) = w_3$; $(m_1; w_2)$ deviates to create ${}_2$.

(ii) ${}_2(m_1) = w_2;\ {}_2(m_2) = w_1;\ {}_1(m_3) = w_3$; $(m_2; w_3)$ deviates to create ${}_3$.

(iii) ${}_3(m_1) = w_2;\ {}_3(m_2) = w_3;\ {}_3(m_3) = w_1$; $(m_1; w_1)$ deviates to create ${}_4$.

Proposition 7. In the oligopolistic competition by joint ventures model, there exists a pairwise stable matching via swapping if = 0 (no externalities: local monopoly).

6 Concluding Remarks

This paper considers stability concepts in one-to-one matching/assignment problems with externalities. We found that the choice of e¤ectiveness functions plays a crucial role in extending the celebrated concept of pairwise stable matching to the environment with

This implies that Y_i is

$$Y_i = {}_iY = \left[1 \quad \frac{(n(\ ) \quad 1)\frac{1}{A(\ )}}{\sum_{j=1}^{n}\frac{1}{A(\ )}}\right]\left[\frac{(n(\ ) \quad 1)}{\sum_{j=1}^{n}\frac{1}{A(\ )}}\right] V$$

These results lead to the following formulas that are essential for the analysis of stability of team structure. Recalling (3), we obtain

$$\begin{aligned}
e_x &= \left[Y_i \quad (1 \quad {}_i)\frac{1}{Y}\right]^{\frac{1}{1-}} a_x^{\overline{1-}} V^{\frac{1}{1-}} \\
&= \left[1 \quad \frac{(n(\ ) \quad 1)\frac{1}{A(\ )}}{\sum_{j=1}^{n}\frac{1}{A(\ )}}\right]\left[\frac{(n(\ ) \quad 1)V}{\sum_{j=1}^{n}\frac{1}{A(\ )}}\right]\left[\frac{(n(\ ) \quad 1)\frac{1}{A(\ )}}{\sum_{j=1}^{n}\frac{1}{A(\ )}} \quad \frac{\sum_{j=1}^{n}\frac{1}{A(\ )}}{(n(\ ) \quad 1)V}\right]^{\frac{1}{1-}} a_x^{\overline{1-}} V^{\frac{1}{1-}} \\
&= \left[1 \quad \frac{(n(\ ) \quad 1)\frac{1}{A(\ )}}{\sum_{j=1}^{n}\frac{1}{A(\ )}}\right]\left[\frac{(n(\ ) \quad 1)\frac{1}{A(\ )}}{\sum_{j=1}^{n}\frac{1}{A(\ )}}\right]\left(\frac{a_x}{A_i(\ )}\right)^{\overline{1-}} V
\end{aligned}$$

This implies that agent i's payoff is written as

$$\begin{aligned}
U_x &= {}_{\text{"}i}V \quad e_x \\
&= \left[1 \quad \frac{(n(\ ) \quad 1)\frac{1}{A(\ )}}{\sum_{j=1}^{n}\frac{1}{A(\ )}}\right] V \quad \left[1 \quad \frac{(n(\ ) \quad 1)\frac{1}{A(\ )}}{\sum_{j=1}^{n}\frac{1}{A(\ )}}\right]\left[\frac{(n(\ ) \quad 1)\frac{1}{A(\ )}}{\sum_{j=1}^{n}\frac{1}{A(\ )}}\right]\left(\frac{a_x}{A_i(\ )}\right)^{\overline{1-}} V \\
&= \left[1 \quad \frac{(n(\ ) \quad 1)\frac{1}{A(\ )}}{\sum_{j=1}^{n}\frac{1}{A(\ )}}\right]\left[1 \quad \frac{(n(\ ) \quad 1)\frac{1}{A(\ )}}{\sum_{j=1}^{n}\frac{1}{A(\ )}}\left(\frac{a_x}{A_i(\ )}\right)^{\overline{1-}}\right] V.
\end{aligned}$$

In order to show that Regularity Condition 1 assures ${}_i > 0$ for all $i = 1; \dots; n$, we use Lemma 1 (i) below. Repeatedly applying Lemma 1 (i), it is easy to see $\sum_{j=1}^{n}\frac{1}{A(\ ^*)} \quad \sum_{j=1}^{n}\frac{1}{A(\ )}$ for all . Moreover, $A_n(\ ) \quad A_i(\ )$ for all . Thus, if the Regularity condition is satisfied, then ${}_i(\ ) > 0$ for all and all $i = 1; \dots; n$. We have completed the proof.□

Proof of Lemma 1. Let $A_j(\ ) \quad a(m_j)^{\overline{1-}} + a(\ (m_j))^{\overline{}}$

Thus,

$$\frac{1}{\tilde{A}_{\ell}(\ ) + \sim^{\frac{1-}{}}} + \frac{1}{\tilde{A}_{k}(\ ) \quad \sim^{\frac{1-}{}}} \qquad \frac{1}{A_{\ell}(\ )} \qquad \frac{1}{A_{k}(\ )}$$

$$= \quad \tilde{A}_{\ell}(\ ) + \sim^{\frac{1-}{}} \quad A_{\ell}(\ )^{\frac{1-}{}} + \quad \tilde{A}_{k}(\ ) \quad \sim^{\frac{1-}{}} \quad \tilde{A}_{k}(\ )^{\frac{1-}{}}$$

Summing up over commodities produces:

$$n \quad \sum_{j=1} x_j \quad n \quad \sum_{j=1} x_j = \quad \sum_{j=1} p_j;$$

or

$$\sum_{j=1} x_j = \frac{1}{1+n}(n \qquad P);$$

where $P = \sum_{j=1}^{n} p_j$. Substituting this back to the f.o.c., we obtain:

$$x_i \quad \frac{}{1+n}(n \qquad P) = p_i;$$

or

$$x_i = x_i(p_i; P) = \qquad p_i \quad \frac{}{1+n}(n \qquad P):$$

Thus, the market demand function for good i is:

$$x_i(p_i; P) = \frac{}{1+n} + \frac{}{1+n} P \qquad p_i:$$

We have completed the proof.■

Proof of Proposition 3. The …rm i's f.o.c. with respect to p_i is:

$$\frac{}{1+n} + \frac{}{1+n} P \qquad p_i \quad + (p_i \qquad c_i) \quad \frac{}{1+n} \qquad 1 \quad = 0;$$

or

$$1 + \frac{1 + (n \quad 1)}{1+n} \quad p_i = \frac{}{1+n} + \frac{}{1+n} P + \frac{1 + (n \quad 1)}{1+n} c_i$$

or

$$\frac{2 + (2n \quad 1)}{1+n} p_i = \frac{}{1+n} + \frac{}{1+n} P + \frac{1 + (n \quad 1)}{1+n} c_i$$

$$p_i = \frac{}{2 + (2n \quad 1)} + \frac{}{2 + (2n \quad 1)} P + \frac{1 + (n \quad 1)}{2 + (2n \quad 1)} c_i$$

Summing them up, we have

$$P = \frac{n}{2 + (2n \quad 1)} + \frac{n}{2 + (2n \quad 1)} P + \frac{1 + (n \quad 1)}{2 + (2n \quad 1)} \sum_{j=1} c_j:$$

Thus,

$$\frac{2 + (n \quad 1)}{2 + (2n \quad 1)} P = \frac{n}{2 + (2n \quad 1)} + \frac{1 + (n \quad 1)}{2 + (2n \quad 1)} \sum_{i=1} c_i;$$

or

$$P = \frac{n}{2 + (n \quad 1)} + \frac{1 + (n \quad 1)}{2 + (n \quad 1)} \sum_{j=1} c_j :$$

Substituting this into the formula for p_i, we obtain

$$\begin{aligned}
p_i &= \frac{}{2 + (2n \quad 1)} + \frac{}{2 + (2n \quad 1)} \left(\frac{n}{2 + (n \quad 1)} + \frac{1 + (n \quad 1)}{2 + (n \quad 1)} \sum_{j=1} c_j \right) + \frac{1 + (n \quad 1)}{2 + (2n \quad 1)} c_i \\
&= \frac{}{2 + (2n \quad 1)} \left(1 + \frac{n}{2 + (n \quad 1)} \right) + \frac{}{2 + (2n \quad 1)} \frac{1 + (n \quad 1)}{2 + (n \quad 1)} \sum_{j=1} c_j + \frac{1 + (n \quad 1)}{2 + (2n \quad 1)} c_i \\
&= \frac{}{2 + (n \quad 1)} + \frac{(1 + (n \quad 1) \ )}{(2 + (2n \quad 1) \ )(2 + (n \quad 1) \ )} \sum_{j=1} c_j + \frac{1 + (n \quad 1)}{2 + (2n \quad 1)} c_i
\end{aligned}$$

Thus, in equilibrium, x_i is

$$\begin{aligned}
x_i &= \frac{}{1 + n} + \frac{}{1 + n} P \quad p_i \\
&= \frac{}{1 + n} + \frac{}{1 + n} \left(\frac{n}{2 + (n \quad 1)} + \frac{1 + (n \quad 1)}{2 + (n \quad 1)} \sum_{j=1} c_j \right) \\
&\quad \left(\frac{}{2 + (n \quad 1)} + \frac{(1 + (n \quad 1) \ )}{(2 + (2n \quad 1) \ )(2 + (n \quad 1) \ )} \sum_{j=1} c_j + \frac{1 + (n \quad 1)}{2 + (2n \quad 1)} c_i \right) \\
&= \frac{\{2 + (n \quad 1) \quad + n \qquad (1 + n \ )\}}{(1 + n \ )(2 + (n \quad 1) \ )} \\
&\quad + \frac{\{(2 + (2n \quad 1) \ )(1 + (n \quad 1) \ ) \quad (1 + n \ )(1 + (n \quad 1) \ )\}}{(1 + n \ )(2 + (2n \quad 1) \ )(2 + (n \quad 1) \ )} \sum_{j=1} c_j \quad \frac{1 + (n \quad 1)}{2 + (2n \quad 1)} c_i \\
&= \frac{1 + (n \quad 1)}{1 + n} \left\{ \frac{}{2 + (n \quad 1)} + \frac{(1 + (n \quad 1) \ )}{(2 + (2n \quad 1) \ )(2 + (n \quad 1) \ )} \sum_{j=1} c_j \quad \frac{1 + n}{2 + (2n \quad 1)} c_i \right\} :
\end{aligned}$$

Then, firm i's equilibrium profit is

$$y_i(\ ) = \frac{1 + (n \quad 1)}{1 + n} \left(\frac{}{2 + (n \quad 1)} + \frac{(1 + (n \quad 1) \ )}{(2 + (2n \quad 1) \ )(2 + (n \quad 1) \ )} \sum_{j=1} c_j \quad \frac{1 + n}{2 + (2n \quad 1)} c_i \right)^2 :$$

From the submodularity of $f(a_m; a_w)$, $\sum_{i=1}^{n} c(m_i; w_i) = \min_{\ \in M} \sum_{i=1}^{n} c(m_i; \ (m_i))$ holds. Thus, Regularity Condition 2 assures that the contents of the parenthesis is positive and $x_i > 0$ holds for all $i = 1; \ldots; n$.■

Proof of Lemma 3. It is easy to see $f(\ _m; a_w) \quad f(a_{m'}; a_w) \quad f(a_m; a_{w'}) \quad f(a_{m'}; a_{w'}) < 0$, since $a_m > a_{m'}$ and $a_w > a_{w'}$, and $\frac{\partial f}{\partial a} < 0$, $\frac{\partial f}{\partial a} < 0$, and $\frac{\partial^2 f}{\partial a \ \partial a} \quad 0$. Thus, $c(m; w) + c(m'; w')$

$c(m; w^0) + c(m^0; w)$ holds. By letting $_{w}$ $c(m^0; w)$ $c(m; w) > 0$ and $_{w'}$ $c(m^0; w^0)$ $c(m; w^0) > 0$, we have $_{w}$ $_{w'}$ 0

$s'_{n-2} < X_{n-1n-2} - (X_{n-1n-1} - (X_{nn-1} - X_{nn}))$. From the previous step, we know $s'_{n-1} \geq s_{n-1}$, and thus $r'_{n-1} \leq X_{n-1n-1} - s_{n-1} = X_{n-1n-1} - (X_{nn-1} - X_{nn})$. Thus, we have

$$
\begin{aligned}
s'_{n-2} + r'_{n-1} &< X_{n-1n-2} - (X_{n-1n-1} - (X_{nn-1} - X_{nn})) + X_{n-1n-1} - (X_{nn-1} - X_{nn}) \\
&= X_{n-1n-2}.
\end{aligned}
$$

This violates the stability, and contradicts with s' being a competitive salary. Thus $s'_{n-2} \geq s_{n-2}$. Repeated applications of the same logic conclude that any competitive salary vector s' satisfies $s' \geq s$. ■

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